

On Hypotheses Testing for Poisson Processes. Regular Case

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Abstract

We consider the problem of hypothesis testing in the situation when the first hypothesis is simple and the second one is local one-sided composite. We describe the choice of thresholds and the power functions of Score-function test, General Likelihood Ratio test, Bayesian tests and Wald test in the situation when the intensity function of inhomogeneous Poisson process is smooth with respect to the parameter. It is shown that almost all these tests are asymptotically uniformly most powerful. The results of numerical simulations are presented.

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1 Introduction

The hypotheses testing theory is well developed branch of the mathematical statistics [12]. The asymptotic approach allows to find satisfactory solutions in many different statements. The simplest problems like the testing of two

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simple hypotheses have well known solution. Recall that if we fix the first type error and seek the test which maximizes the power, then we obtain immediately (by Neyman-Pearson Lemma) the most powerful test based on the likelihood ratio statistic. The case of composite alternative is more difficult to treat and here the asymptotic solution is available in the regular case. It is possible, using, for example, the score-function test (SFT) to construct the asymptotically (locally) most powerful test. Moreover, the General Likelihood Ratio Test (GLRT) and the Wald test (WT), based on the maximum likelihood estimator, are asymptotically most powerful in the same sense. In the non regular cases the situation became much more complicate. First of all there are different non regular (singular) situations and in all these situations the choice of the asymptotically the best test is always an open question.

This work is an attempt to study all these situations on the model of inhomogeneous Poisson processes of intensity function $\lambda(t), 0 \leq t \leq \tau$. This model is sufficiently simple to allow us to realize first the well known tests in the regular case and to verify that for this model too the construction of the asymptotically most powerful tests (SFT, GLRT, WT) is possible. In the next paper we study the behavior of these tests in the case of singular statistical models. The “evolution of singularity” of the intensity function is the following: regular (finite Fisher information, this paper), continuous but not differentiable (cusp-type singularity), discontinuous intensity function [4]. In all three cases we describe analytically the tests. This means that we describe the test statistics, the choice of thresholds and the form of the power functions for local alternatives.

Note that the notion of *local alternative* is different following the type of regularity-singularity. In the regular case and the simple hypothesis $\vartheta = \vartheta_1$ against one-sided alternative $\vartheta > \vartheta_1$, the local alternative can be $\vartheta = \vartheta_1 + \frac{u}{\sqrt{n}}$, $u > 0$. In the cusp-type singularity, the local alternative is $\vartheta = \vartheta_1 + \frac{u}{n^{\frac{1}{2\kappa+1}}}$, $u > 0$ and in the case of discontinuous intensity function we put $\vartheta = \vartheta_1 + \frac{u}{n}$, $u > 0$. In all these problems the most interesting for us question is the comparison of the power functions of different tests. In singular situations these comparison is done with the help of numerical simulations. The main results concern the limit likelihood ratios in non-regular situations and the same limits have likelihood ratios in the many other models of observations (i.i.d., time series, diffusion processes etc.) see, e.g., [6], [2]. Therefore the presented here results are of more universal nature and are valid for any other model (non Poissonian) with the same limit likelihood ratios.

We recall that $X = (X_t, t \geq 0); X_0 = 0$ is an inhomogeneous Poisson process with intensity function $\lambda(t)$, if $X_0 = 0$, the increments of X on

disjoint intervals are independent and distributed according to the Poisson law

$$\mathbf{P}\{X_t - X_s = k\} = \frac{\left(\int_s^t \lambda(t) dt\right)^k}{k!} \exp\left\{-\int_s^t \lambda(t) dt\right\}.$$

All statistical problems considered in this work concerned the intensity functions depending on some one-dimensional parameter, i.e., $\lambda(t) = \lambda(\vartheta, t)$. The basic hypothesis is always the same : $\vartheta = \vartheta_1$ and the alternative $\vartheta > \vartheta_1$. The diversity of the statements corresponds to the different types of regularity of the function $\lambda(\vartheta, t)$. The case of unknown period ϑ needs a special study.

The hypotheses testing problems (or closely related properties of the likelihood ratio) for inhomogeneous Poisson processes were studied by many authors, see, for example, Brown [1], Kutoyants [7], Léger and Wolfson [11], Liese and Lorz [14], Sung *et al.* [16], Fazli and Kutoyants [5], Dachian and Kutoyants [3] and the references therein. Note that the results of this study will appear later in the work [9].

2 Auxiliary results

For simplicity of exposition we consider the model of n independent observations of inhomogeneous Poisson processes $X^n = (X_1, \dots, X_n)$, where $X_j = \{X_j(t), 0 \leq t \leq \tau\}$. We have

$$\mathbf{E}_\vartheta X_j(t) = \Lambda(\vartheta, t) = \int_0^t \lambda(\vartheta, s) ds.$$

Here ϑ is one-dimensional parameter and \mathbf{E}_ϑ is the mathematical expectation, when the true value is $\vartheta \in \Theta = [\vartheta_1, b], b < \infty$. Note that this model is equivalent to another one of observation of inhomogeneous Poisson processes $X^T = [X_t, 0 \leq t \leq T]$ with periodic intensity $\lambda(\vartheta, t + j\tau) = \lambda(\vartheta, t)$, $j = 1, 2, \dots, n$ and $T = n\tau$ (the period τ is supposed to be known). Indeed, if we put $X_j(s) = X_{s+\tau(j-1)} - X_{\tau(j-1)}$, $s \in [0, \tau], j = 1, \dots, n$, then the observation of one trajectory X^T is equivalent to n independent observations X_1, \dots, X_n .

Therefore, we suppose that we observe n copies of inhomogeneous Poisson process $X^n = (X_1, \dots, X_n)$ with the intensity function $\lambda(\vartheta, t), 0 \leq t \leq \tau$. The intensity function is supposed to be separated from zero on $[0, \tau]$, the measures corresponding to Poisson processes with the different values of ϑ are equivalent. The likelihood function is defined by the equality (see Liese

[13])

$$L(\vartheta, X^n) = \exp \left\{ \sum_{j=1}^n \int_0^\tau \ln \lambda(\vartheta, t) dX_j(t) - n \int_0^\tau [\lambda(\vartheta, t) - 1] dt \right\}$$

and the likelihood ratio function is

$$L(\vartheta, \vartheta_1, X^n) = L(\vartheta, X^n) / L(\vartheta_1, X^n).$$

We have to test the following two hypotheses

$$\begin{aligned} \mathcal{H}_1 & : \quad \vartheta = \vartheta_1, \\ \mathcal{H}_2 & : \quad \vartheta > \vartheta_1. \end{aligned}$$

We define a test $\bar{\psi}_n = \bar{\psi}_n(X^n)$ as the probability to accept the hypothesis \mathcal{H}_2 . The power function is $\beta(\bar{\psi}_n, \vartheta) = \mathbf{E}_\vartheta \bar{\psi}_n(X^n)$, $\vartheta > \vartheta_1$.

Denote by \mathcal{K}_ε the class of test functions $\bar{\psi}_n$ of asymptotic size $\varepsilon \in [0, 1]$

$$\mathcal{K}_\varepsilon = \left\{ \bar{\psi}_n : \lim_{n \rightarrow \infty} \mathbf{E}_{\vartheta_1} \bar{\psi}_n(X^n) = \varepsilon \right\}.$$

Our goal is to construct the tests which belong to this class and have some properties of asymptotic optimality. The comparison of tests can be done by comparison of their power functions. It is known that for any reasonable test and for any fixed alternative the power function tends to 1. To avoid this difficulty as usual we consider *close* or *contigual* alternatives. Let us put $\vartheta = \vartheta_1 + \varphi_n u$, where $u \in U_n^+ = [0, \varphi_n^{-1}(b - \vartheta_1)]$, $\varphi_n = \varphi_n(\vartheta_1) > 0$ and $\varphi_n \rightarrow 0$. The rate of convergence $\varphi_n \rightarrow 0$ is such that the normalized likelihood ratio

$$Z_n(u) = \frac{L(\vartheta_1 + \varphi_n u, X^n)}{L(\vartheta_1, X^n)}, \quad u \geq 0$$

has non degenerate limit. In the regular case this rate is usually $\varphi_n = n^{-1/2}$.

Then the initial problem of hypotheses testing can be rewritten as

$$\begin{aligned} \mathcal{H}_1 & : \quad u = 0, \\ \mathcal{H}_2 & : \quad u > 0. \end{aligned}$$

The corresponding power function of the test $\bar{\psi}_n$ is denoted as

$$\beta_n(\bar{\psi}_n, u) = \mathbf{E}_{\vartheta_1 + \varphi_n u} \bar{\psi}_n, \quad u > 0.$$

We introduce the asymptotic optimality of tests with the help of the following definition (see [15]).

Definition 1. We call a test $\psi_n^*(X^n) \in \mathcal{K}_\varepsilon$ locally asymptotically uniformly most powerful (LAUMP) in the class \mathcal{K}_ε if its power function $\beta_n(\psi_n^*, u)$ satisfies the relation: for any test $\bar{\psi}_n(X^n) \in \mathcal{K}_\varepsilon$ and any $K > 0$ we have

$$\lim_{n \rightarrow \infty} \inf_{0 < u \leq K} [\beta_n(\psi_n^*, u) - \beta_n(\bar{\psi}_n, u)] \geq 0. \quad (1)$$

Below we show that in the regular case many tests can be LAUMP. In the next paper [4], where we consider some singular situations the definition of the reasonable asymptotic optimality of tests is an open question. That is why to compare the same tests we turn to the methods of numerical simulations.

We assume that the following *Regularity conditions* are satisfied.

Smoothness. The intensity function $\lambda(\vartheta, t), 0 \leq t \leq \tau$ of the observed Poisson process X^n is two times continuously differentiable w.r.t. ϑ , separated from zero uniformly on $\vartheta \geq \vartheta_1$ and the Fisher information is positive:

$$I(\vartheta) = \int_0^\tau \frac{\dot{\lambda}(\vartheta, t)^2}{\lambda(\vartheta, t)} dt, \quad \inf_{\vartheta \in \Theta} I(\vartheta) > 0.$$

Distinguishability. For any $\nu > 0$

$$\inf_{\vartheta_* \in \Theta} \inf_{|\vartheta - \vartheta_*| > \nu} \left\| \sqrt{\lambda(\vartheta, \cdot)} - \sqrt{\lambda(\vartheta_1, \cdot)} \right\|_{L^2} > 0.$$

Here point means derivative w.r.t. ϑ and

$$\|h(\cdot)\|_{L^2}^2 = \int_0^\tau h(t)^2 dt.$$

In this case the natural normalization function is $\varphi_n = n^{-1/2}$ and the change of variables is $\vartheta = \vartheta_1 + \frac{u}{\sqrt{n}}$. The key propriety of the statistical problems in regular case is the *local asymptotic normality* (LAN) of the family of measures of corresponding inhomogeneous Poisson processes at the point ϑ_1 .

This means that the normalized likelihood-ratio

$$\tilde{Z}_n(u) = L\left(\vartheta_1 + \frac{u}{\sqrt{n}}, \vartheta_1, X^n\right)$$

admits the representation

$$\tilde{Z}_n(u) = \exp \left\{ u \tilde{\Delta}_n(\vartheta_1, X^n) - \frac{u^2}{2} I(\vartheta_1) + r_n \right\},$$

where by the central limit theorem, we have

$$\tilde{\Delta}_n(\vartheta_1, X^n) = \frac{1}{\sqrt{n}} \sum_{j=1}^n \int_0^\tau \frac{\dot{\lambda}(\vartheta_1, t)}{\lambda(\vartheta_1, t)} [dX_j(t) - \lambda(\vartheta_1, t) dt] \implies \tilde{\Delta}$$

with $\tilde{\Delta} \sim \mathcal{N}(0, I(\vartheta_1))$ and $r_n = r_n(\vartheta_1, u, X^n) \xrightarrow{p} 0$. Moreover, the convergence is uniform on $0 \leq u < K$ for any $K > 0$.

Let us recall how this representation was obtained [7]. Denoting $\lambda_0 = \lambda(\vartheta_1, t)$ and $\lambda_u = \lambda\left(\vartheta_1 + \frac{u}{\sqrt{n}}, t\right)$, with the help of the Taylor series expansion we can write

$$\begin{aligned} \ln Z_n(u) &= \sum_{j=1}^n \int_0^\tau \ln \frac{\lambda_u}{\lambda_0} [dX_j(t) - \lambda_0 dt] - n \int_0^\tau \left[\lambda_u - \lambda_0 - \lambda_0 \ln \frac{\lambda_u}{\lambda_0} \right] dt \\ &= \frac{u}{\sqrt{n}} \sum_{j=1}^n \int_0^\tau \frac{\dot{\lambda}_0}{\lambda_0} [dX_j(t) - \lambda_0 dt] - \frac{u^2}{2} \int_0^\tau \frac{\dot{\lambda}_0^2}{\lambda_0} dt + r_n \\ &= u \tilde{\Delta}_n(\vartheta_1, X^n) - \frac{u^2}{2} I(\vartheta_1) + r_n \implies \tilde{\Delta} - \frac{u^2}{2} I(\vartheta_1). \end{aligned}$$

Here and in the sequel we choose the reparametrization which leads to *universal* in some sense limits. For example, in regular case we can put

$$\varphi_n = \varphi_n(\vartheta_1) = \frac{1}{\sqrt{nI(\vartheta_1)}}, \quad u \in \mathbb{U}_n^+ = [0, \varphi_n^{-1}(b - \vartheta_1)].$$

With such change of variables the

$$\Delta_n(\vartheta_1, X^n) = \frac{1}{\sqrt{I(\vartheta_1)}} \tilde{\Delta}_n \implies \Delta \sim \mathcal{N}(0, 1).$$

and also

$$Z_n(u) = L(\vartheta_1 + u\varphi_n, \vartheta_1, X^n) = \exp \left\{ u \Delta_n(\vartheta_1, X^n) - \frac{u^2}{2} + r_n \right\}.$$

The LAN families have many remarkable properties and some of them we will use below.

Let us remind here one general result which is valid for the wider class of distributions. We suppose only that the normalized likelihood-ratio $Z_n(u)$ converges to some limit $Z(u)$ in distribution. Such situations we have in all our regular and singular problems. This property allow us to calculate the distribution under local alternative if we know the distribution under the null hypothesis. Moreover, it gives an efficient algorithm for the calculation of the power functions in the numerical simulations.

Lemma 1. (Le Cam's Third Lemma) *Suppose that $(Z_n(u), Y_n)$ converges in distribution under measure $\mathbf{P}_{\vartheta_1}^{(n)}$:*

$$(Z_n(u), Y_n) \Longrightarrow (Z(u), Y).$$

Then for any bounded continuous function $g(\cdot)$

$$\mathbf{E}_{\vartheta_1 + \varphi_n u} [g(Y_n)] \longrightarrow \mathbf{E} [Z(u) g(Y)].$$

For the proof see [10].

In the regular case the limit of $Z_n(u)$ is the random function

$$Z(u) = \exp \left\{ u \Delta - \frac{u^2}{2} \right\}, \quad u \geq 0,$$

i.e., we have (for any fixed $u > 0$) the convergence

$$Z_n(u) \Longrightarrow Z(u).$$

According to this lemma we can write for characteristic function of $\Delta_n = \Delta_n(\vartheta_1, X^n)$ the following relations

$$\mathbf{E}_{\vartheta_1 + \varphi_n u} e^{i\mu \Delta_n} \rightarrow \mathbf{E} Z(u) e^{i\mu u \Delta} = e^{-\frac{u^2}{2}} \mathbf{E} e^{u\Delta + i\mu \Delta} = e^{i\mu u - \frac{\mu^2}{2}} = \mathbf{E} e^{i\mu(u + \Delta)},$$

which yields the distribution of the statistics Δ_n under alternative

$$\Delta_n(\vartheta_1, X^n) \Longrightarrow u + \Delta \sim \mathcal{N}(u, 1).$$

3 Weak convergence

All tests which we study are functionals of the normalized likelihood-ratio $Z_n(\cdot)$. For each test we have to evaluate two quantities. The first one is the threshold which provides asymptotically the guaranteed type one error and the second is the power function, which has to be calculated under alternative. Our study is based on the weak convergence of the likelihood ratio $Z_n(\cdot)$ under hypothesis (to calculate the threshold) and under alternative (to calculate the limit power function). Note that the test statistics of all tests are continuous functionals of $Z_n(\cdot)$, that is why we verify the weak convergence of $Z_n(\cdot)$ under hypothesis and under alternative and these allow us to obtain the limit distributions of the statistics.

The observed inhomogeneous Poisson processes X^n has the distribution $\mathbf{P}_{\vartheta}^{(n)}$ induced on the measurable space of its realizations. The measures of the

family $\{\mathbf{P}_{\vartheta}^{(n)}, \vartheta \geq \vartheta_1\}$ are equivalent. Introduce the normalized likelihood ratio

$$\begin{aligned} \ln Z_n(u) &= \sum_{j=1}^n \int_0^\tau \ln \frac{\lambda(\vartheta + \varphi_n(\vartheta)u, t)}{\lambda(\vartheta, y)} dX_j(t) \\ &\quad - n \int_0^\tau [\lambda(\vartheta + \varphi_n(\vartheta)u, t) - \lambda(\vartheta, t)] dt, \end{aligned}$$

where $u \in \mathbb{U}_n = [\varphi_n^{-1}(\vartheta_1 - \vartheta), \varphi_n^{-1}(b - \vartheta)]$. We define $Z_n(u)$ linearly decreasing to zero on the intervals $[\varphi_n^{-1}(b - \vartheta), \varphi_n^{-1}(b - \vartheta) + 1]$ and similarly on the interval $[\varphi_n^{-1}(\vartheta_1 - \vartheta) - 1, \varphi_n^{-1}(\vartheta_1 - \vartheta)]$. Outside we put $Z_n(u) \equiv 0$. Now the random function $Z_n(u)$ is defined on \mathcal{R} and belongs to the space $\mathcal{C}_0(\mathcal{R})$ of continuous on \mathcal{R} functions such that $z(u) \rightarrow 0$ as $|u| \rightarrow \infty$. Introduce the uniform metric in this space and denote by \mathcal{B} the corresponding borelian sigma-algebra. The next theorem describes the weak convergence in the measurable space $(\mathcal{C}_0(\mathcal{R}), \mathcal{B})$ of the random processes $Z_n(v), v \in \mathcal{R}$ under alternative $\vartheta = \vartheta_1 + \varphi_n u_*$ with fixed $u_* > 0$ to the random process $Z(v, u_*) = \exp\left\{v\Delta + vu_* - \frac{u_*^2}{2}\right\}, v \in \mathcal{R}$. Note that in [8] this theorem was proved for a “fixed true value ϑ ”. In the hypotheses testing problems considered here we need this convergence the first time (under hypothesis \mathcal{H}_0) for fixed $\vartheta = \vartheta_1$ ($u_* = 0$) and the second time for the alternative with “moving true value” $\vartheta_{u_*} = \vartheta_1 + \varphi_n u_*$.

Theorem 1. *Let us suppose that the Regularity conditions are fulfilled. Then we have the weak convergence of the random process $Z_n = (Z_n(v), v \geq 0)$ to $Z = (Z(v, u_*), v \geq 0)$.*

According to [6, Theorem 1.10.1] to prove this theorem we have to verify the following properties of the process $Z_n(\cdot)$.

1. The finite-dimensional distributions of $Z_n(\cdot)$ converge to the finite-dimensional distributions of $Z(\cdot, u_*)$.
2. The inequality

$$\mathbf{E}_{\vartheta_{u_*}} \left| Z_n^{\frac{1}{2}}(v_2) - Z_n^{\frac{1}{2}}(v_1) \right|^2 \leq C |v_2 - v_1|^2$$

holds for every $v_1, v_2 \in \mathbb{U}_n^+$ and some constant $C > 0$.

3. There exists $d > 0$, such that for some $n_0 > 0$ and all $n \geq n_0$ we have the estimate

$$\mathbf{P}_{\vartheta_{u_*}} \left\{ Z_n(v) > e^{-d|v - u_*|^2} \right\} \leq e^{-d|v - u_*|^2}.$$

Lemma 2. *The finite-dimensional distributions of $Z_n(\cdot)$ converge to the finite-dimensional distributions of $Z(\cdot, u_*)$.*

Let us write the random function $Z_n(v)$ under alternative $\vartheta = \vartheta_1 + u_*\varphi_n$ as follows:

$$\begin{aligned} Z_n(v) &= L(\vartheta_1 + v\varphi_n, \vartheta_1, X^n) \\ &= L(\vartheta_1 + u_*\varphi_n, \vartheta_1, X^n) L(\vartheta_1 + v\varphi_n, \vartheta_1 + u_*\varphi_n, X^n). \end{aligned}$$

For the first term we have

$$L(\vartheta_1 + u_*\varphi_n, \vartheta_1, X^n) \implies \exp\left\{u_*\Delta + \frac{u_*^2}{2}\right\}.$$

Therefore we only need to check the conditions 1-3 for the term

$$Z_n(v, u_*) = L(\vartheta_1 + v\varphi_n, \vartheta_1 + u_*\varphi_n, X^n).$$

The limit process for $Z_n(v, u_*)$ is

$$Z(v, u_*) = \exp\left\{(v - u_*)\Delta - \frac{(v - u_*)^2}{2}\right\}, \quad v \in \mathcal{R}.$$

Hence

$$Z_n(v) \implies Z(v, u_*) = \exp\left\{v\Delta + vu_* - \frac{v^2}{2}\right\}.$$

For the details see, e.g., [8] in the similar situation.

Lemma 3. *Let the Regularity conditions be fulfilled. Then there exists a constant $C > 0$, such that*

$$\mathbf{E}_{\vartheta_1 + u_*\varphi_n} |Z_n^{1/2}(v_1, u_*) - Z_n^{1/2}(v_2, u_*)|^2 \leq C |v_1 - v_2|^2$$

for all $v_1, v_2 \in \mathbb{U}_n^+$ and sufficiently large values of n .

Proof. According to [8, Lemma 1.1.5], we have, for $v_1 > v_2 > 0$ (the other cases can be treated in the similar way),

$$\begin{aligned} &\mathbf{E}_{\vartheta_1 + \varphi_n u_*} |Z_n^{1/2}(v_1, u_*) - Z_n^{1/2}(v_2, u_*)|^2 \\ &\leq n \int_0^\tau \left(\frac{\lambda^{1/2}(\vartheta_1 + v_1\varphi_n, t)}{\lambda^{1/2}(\vartheta_1 + u_*\varphi_n, t)} - \frac{\lambda^{1/2}(\vartheta_1 + v_2\varphi_n, t)}{\lambda^{1/2}(\vartheta_1 + u_*\varphi_n, t)} \right)^2 \lambda(\vartheta_1 + u_*\varphi_n, t) dt \\ &= n \int_0^\tau (\lambda^{1/2}(\vartheta_1 + v_1\varphi_n, t) - \lambda^{1/2}(\vartheta_1 + v_2\varphi_n, t))^2 dt \\ &= \frac{n}{4} \varphi_n^2 (v_2 - v_1)^2 \int_0^\tau \frac{\dot{\lambda}(\tilde{\vartheta}_v, t)^2}{\lambda(\vartheta_1, t)} dt \leq C (v_2 - v_1)^2. \end{aligned}$$

□

Lemma 4. *Let the Regularity conditions be fulfilled. Then there exists a constant $d > 0$, such that*

$$\mathbf{P}_{\vartheta_1 + u_* \varphi_n} \left\{ Z_n(v) > e^{-d|v - u_*|^2} \right\} \leq e^{-d|v - u_*|^2} \quad (2)$$

for all $u_*, v \in \mathbb{U}_n^+$ and sufficiently large value of n .

Proof. Using the Markov inequality, we get

$$\mathbf{P}_{\vartheta_1 + u_* \varphi_n} \left\{ Z_n(v) > e^{-d|v - u_*|^2} \right\} \leq e^{\frac{1}{2}d|v - u_*|^2} \mathbf{E}_{\vartheta_1 + u_* \varphi_n} Z_n^{1/2}(v).$$

According to [8, Lemma 1.1.5], we have,

$$\begin{aligned} & \mathbf{E}_{\vartheta_1 + u_* \varphi_n} Z_n^{1/2}(v, u_*) \\ &= \exp \left\{ -\frac{1}{2} \int_0^{n\tau} \left(\frac{\lambda^{1/2}(\vartheta_1 + v\varphi_n, t)}{\lambda^{1/2}(\vartheta_1 + u_*\varphi_n, t)} - 1 \right)^2 \lambda(\vartheta_1 + u_*\varphi_n, t) dt \right\} \\ &= \exp \left\{ -\frac{1}{2} n \int_0^\tau \left(\lambda^{1/2}(\vartheta_1 + v\varphi_n, t) - \lambda^{1/2}(\vartheta_1 + u_*\varphi_n, t) \right)^2 dt \right\}, \end{aligned}$$

Using the Taylor's expansion we get

$$\lambda^{1/2}(\vartheta_1 + v\varphi_n, t) = \lambda^{1/2}(\vartheta_1 + u_*\varphi_n, t) + \frac{\varphi_n(v - u_*)}{2} \frac{\dot{\lambda}(\tilde{\vartheta}, t)}{\lambda^{1/2}(\tilde{\vartheta}, t)}.$$

Hence, for sufficiently large n providing $|v - u_*| \varphi_n \leq \gamma$ we have

$\mathbf{I}(\tilde{\vartheta}) \geq \frac{1}{2} \mathbf{I}(\vartheta_1)$, and we obtain

$$\mathbf{E}_{\vartheta_1 + u_* \varphi_n} Z_n^{1/2}(v, u_*) \leq \exp \left\{ -\frac{1}{8 \mathbf{I}(\vartheta_1)} |v - u_*|^2 \mathbf{I}(\tilde{\vartheta}) \right\} \leq \exp \left\{ -\frac{|v - u_*|^2}{16} \right\}. \quad (3)$$

By Distinguishability condition, we can write

$$g(\gamma) = \inf_{\varphi_n |v - u| > \gamma} \int_0^\tau \left(\lambda^{1/2}(\vartheta_1 + v\varphi_n, t) - \lambda^{1/2}(\vartheta_1 + u_*\varphi_n, t) \right)^2 dt > 0$$

and hence

$$\int_0^\tau \left(\lambda^{1/2}(\vartheta_1 + v\varphi_n, t) - \lambda^{1/2}(\vartheta_1 + u_*\varphi_n, t) \right)^2 dt \geq g(\gamma) \geq g(\gamma) \frac{\varphi_n^2(u_* - v)^2}{(b - \vartheta_1)^2}$$

and

$$\mathbf{E}_{\vartheta_1 + u_* \varphi_n} Z_n^{1/2}(v) \leq \exp \left\{ -\frac{g(\gamma) |v - u_*|^2}{2\mathbf{I}(\vartheta_1) (b - \vartheta_1)^2} \right\}. \quad (4)$$

Let us put

$$d = \frac{2}{3} \min \left(\frac{1}{16}, \frac{g(\gamma)}{2\mathbf{I}(\vartheta_1) (b - \vartheta_1)^2} \right).$$

Then the estimate (2) follows from (3) and (4). \square

The weak convergence of $Z_n(\cdot, u_*)$ now follows from [6, Theorem 1.10.1].

4 Hypothesis testing

We consider the construction of score function test, general likelihood ratio test, Wald test and two bayesian tests. For all tests we describe the choice of the thresholds and evaluate the limit power functions for local alternatives.

4.1 Score function test

Let us introduce *score function test* (SFT)

$$\hat{\psi}_n(X^n) = \mathbb{1}_{\{\Delta_n(\vartheta_1, X^n) > z_\varepsilon\}},$$

where z_ε is the $(1 - \varepsilon)$ -quantile of the standard normal distribution $\mathcal{N}(0, 1)$ and the statistic $\Delta_n(\vartheta_1, X^n)$ is

$$\Delta_n(\vartheta_1, X^n) = \frac{1}{\sqrt{n\mathbf{I}(\vartheta_1)}} \sum_{j=1}^n \int_0^\tau \frac{\dot{\lambda}(\vartheta_1, t)}{\lambda(\vartheta_1, t)} [dX_j(t) - \lambda(\vartheta_1, t) dt].$$

The SFT has the following properties.

Proposition 1. *The test $\hat{\psi}_n(X^n) \in \mathcal{K}_\varepsilon$ and is LAUMP. Its power function*

$$\beta_n(\hat{\psi}_n, u) \longrightarrow \beta^*(u_*) = \mathbf{P}(\Delta > z_\varepsilon - u_*), \quad \Delta \sim \mathcal{N}(0, 1). \quad (5)$$

Proof. The property $\hat{\psi}_n(X^n) \in \mathcal{K}_\varepsilon$ follows immediately from the asymptotic normality

$$\Delta_n(\vartheta_1, X^n) \Longrightarrow \Delta.$$

Further, we have (under alternative $\vartheta_u = \vartheta_1 + u_* \varphi_n$) the convergence

$$\beta_n(\hat{\psi}_n, u_*) \longrightarrow \mathbf{P}(\Delta + u_* > z_\varepsilon) = \beta^*(u_*).$$

This follows from the Third Le Cam's Lemma and can be shown directly as follows. Suppose the intensity of the observed Poisson process is $\lambda(\vartheta_1 + u_*\varphi_n, t)$, then we can write

$$\begin{aligned}\Delta_n(\vartheta_1, X^n) &= \frac{1}{\sqrt{nI(\vartheta_1)}} \sum_{j=1}^n \int_0^\tau \frac{\dot{\lambda}(\vartheta_1, t)}{\lambda(\vartheta_1, t)} [dX_j(t) - \lambda(\vartheta_1 + u_*\varphi_n, t) dt] \\ &\quad + \frac{1}{\sqrt{nI(\vartheta_1)}} \sum_{j=1}^n \int_0^\tau \frac{\dot{\lambda}(\vartheta_1, t)}{\lambda(\vartheta_1, t)} [\lambda(\vartheta_1 + u_*\varphi_n, t) - \lambda(\vartheta_1, t)] dt \\ &= \Delta_n^*(\vartheta_1, X^n) + \frac{u_*}{nI(\vartheta_1)} \sum_{j=1}^n \int_0^\tau \frac{\dot{\lambda}(\vartheta_1, t)^2}{\lambda(\vartheta_1, t)} dt + o(1) \\ &= \Delta_n^*(\vartheta_1, X^n) + u_* + o(1) \implies \Delta + u_*.\end{aligned}$$

To show that the SFT is LAUMP we verify that the limit of its power function coincides with the limit of the power of likelihood ratio (Neyman-Person) test (LRT) $\psi_n^*(X^n)$. Remind that the LRT is the most powerful for each fixed (simple) alternative. Of course, the LRT is not indeed a test because for its construction we use the knowledge of the value of parameter u_* under alternative.

The LRT is defined by

$$\psi_n^*(X^n) = \mathbb{I}_{\{Z_n(u_*) > d_\varepsilon\}} + q_\varepsilon \mathbb{I}_{\{Z_n(u_*) = d_\varepsilon\}},$$

where the threshold d_ε and probability q_ε are chosen from the condition $\psi_n^*(X^n) \in \mathcal{K}_\varepsilon$, i.e.,

$$\mathbf{P}_{\vartheta_1} \{Z_n(u_*) > d_\varepsilon\} + q_\varepsilon \mathbf{P}_{\vartheta_1} \{Z_n(u_*) = d_\varepsilon\} = \varepsilon.$$

Of course, we can put $q_\varepsilon = 0$ because the limit random variable $Z(u_*)$ has continuous distribution function.

The threshold d_ε can be found as follows. The LAN of the family of measures at the point ϑ_1 allows us to write

$$\begin{aligned}\mathbf{P}_{\vartheta_1} (Z_n(u_*) > d_\varepsilon) &= \mathbf{P}_{\vartheta_1} \left(u_* \Delta_n(\vartheta_1, X^n) - \frac{u_*^2}{2} + r_n > \ln d_\varepsilon \right) \\ &\longrightarrow \mathbf{P} \left(u_* \Delta - \frac{u_*^2}{2} > \ln d_\varepsilon \right) = \mathbf{P} \left(\Delta > \frac{\ln d_\varepsilon}{u_*} + \frac{u_*}{2} \right) = \varepsilon.\end{aligned}$$

Hence we have

$$\frac{\ln d_\varepsilon}{u_*} + \frac{u_*}{2} = z_\varepsilon \quad \text{and} \quad d_\varepsilon = \exp \left\{ u_* z_\varepsilon - \frac{u_*^2}{2} \right\}.$$

Therefore the test

$$\psi_n^*(X^n) = \mathbb{I}_{\left\{Z_n(u_*) > \exp\left\{u_* z_\varepsilon - \frac{u_*^2}{2}\right\}\right\}}$$

belongs to \mathcal{K}_ε .

For the power function of this test we have (below $\vartheta_{u_*} = \vartheta_1 + u_* \varphi_n$)

$$\begin{aligned} \beta_n(\psi_n^*, u_*) &= \mathbf{P}_{\vartheta_{u_*}}(Z_n(u_*) > d_\varepsilon) = \mathbf{P}_{\vartheta_{u_*}}(u_* \Delta_n(\vartheta_1, X^n) + r_n > u_* z_\varepsilon) \\ &= \mathbf{P}_{\vartheta_{u_*}}\left(\Delta_n(\vartheta_1, X^n) + \frac{r_n}{u_*} > z_\varepsilon\right) \longrightarrow \mathbf{P}(\Delta + u_* > z_\varepsilon) = \beta^*(u_*). \end{aligned}$$

Therefore the limits of these two tests coincide and the score-function test is asymptotically as good as the Neyman-Pearson optimal one. Note that the limits are valid for any sequence of $0 \leq u_* \leq K$ and for any $K > 0$ and we can choose a sequence $\hat{u}_n \in [0, K]$ such that

$$\sup_{0 \leq u_* \leq K} \left| \beta_n(\psi_n^*, u_*) - \beta_n(\hat{\psi}_n, u_*) \right| = \left| \beta_n(\psi_n^*, \hat{u}_n) - \beta_n(\hat{\psi}_n, \hat{u}_n) \right| \rightarrow 0$$

in obvious notations, which represents the asymptotic coincidence of two tests. □

4.2 GLRT and Wald test

Let us remind the definition of the MLE $\hat{\vartheta}_n$:

$$L(\hat{\vartheta}_n, \vartheta_1, X^n) = \sup_{\vartheta \in [\vartheta_1, b]} L(\vartheta, \vartheta_1, X^n),$$

where the likelihood-ratio function is

$$\begin{aligned} L(\vartheta, \vartheta_1, X^n) &= \exp \left\{ \sum_{j=1}^n \int_0^\tau \ln \frac{\lambda(\vartheta, t)}{\lambda(\vartheta_1, t)} dX_j(t) \right. \\ &\quad \left. - n \int_0^\tau [\lambda(\vartheta, t) - \lambda(\vartheta_1, t)] dt \right\}, \quad \vartheta \in [\vartheta_1, b]. \end{aligned}$$

The GLRT is

$$\bar{\psi}_n(X^n) = \mathbb{I}_{\{Q(X^n) > h_\varepsilon\}}, \quad h_\varepsilon = \exp\{z_\varepsilon^2/2\},$$

where

$$Q(X^n) = \sup_{\vartheta \in [\vartheta_1, b]} L(\vartheta, \vartheta_1, X^n) = L(\hat{\vartheta}_n, \vartheta_1, X^n).$$

The Wald's test is based on the maximum likelihood estimator $\hat{\vartheta}_n$ and is defined as follows

$$\psi_n^o(X^n) = \mathbb{I}_{\{\varphi_n^{-1}(\hat{\vartheta}_n - \vartheta_1) > z_\varepsilon\}}.$$

The properties of these tests are given in the following Proposition.

Proposition 2. *The tests $\bar{\psi}_n(X^n), \psi_n^o(X^n) \in \mathcal{K}_\varepsilon$, their power functions $\beta(\bar{\psi}_n, u)$ and $\beta(\psi_n^o, u)$ converge to $\beta^*(u)$ and therefore the tests are LAUMP.*

Proof. Let us put $\vartheta = \vartheta_1 + v\varphi_n$ and denote $\hat{v}_n = \varphi_n^{-1}(\hat{\vartheta}_n - \vartheta_1)$. We have

$$\begin{aligned} \mathbf{P}_{\vartheta_1} \left\{ \sup_{\vartheta \in [\vartheta_1, b]} L(\vartheta, \vartheta_1, X^n) > h_\varepsilon \right\} &= \mathbf{P}_{\vartheta_1} \left\{ \sup_{v \in \mathbb{U}_n^+} L(\vartheta_1 + v\varphi_n, \vartheta_1, X^n) > h_\varepsilon \right\} \\ &= \mathbf{P}_{\vartheta_1} \left\{ \sup_{v \in \mathbb{U}_n^+} Z_n(v) > h_\varepsilon \right\}. \end{aligned}$$

The weak convergence of the random function $Z_n(v), v \in \mathbb{U}_n^+$ follows from the Theorem 1, where we have to put $u_* = 0$.

Therefore, we have the weak convergence of the measures of the random processes $\{Z_n^{1/2}(v), v \geq 0\}$ to the measure of the process $\{Z^{1/2}(v), v \geq 0\}$ at the point ϑ_1 . This provides us the convergence of the distributions of all continuous in uniform metric functionals. Hence

$$\begin{aligned} Q(X^n) = \sup_{v > 0} Z_n(v) &\implies \sup_{v > 0} Z(v) \\ &= \sup_{v > 0} \exp \left\{ v\Delta - \frac{v^2}{2} \right\} = \exp \left\{ \frac{\Delta^2}{2} \mathbb{I}_{\{\Delta \geq 0\}} \right\}. \end{aligned}$$

This provides the convergence (we suppose that $\varepsilon \leq \frac{1}{2}$)

$$\mathbf{E}_{\vartheta_1} \hat{\psi}_n(X^n) \longrightarrow \mathbf{P}\{\Delta > z_\varepsilon\} = \varepsilon.$$

Remind that for $\varepsilon < \frac{1}{2}$

$$\mathbf{P}\{\Delta \mathbb{I}_{\{\Delta \geq 0\}} > z_\varepsilon\} = \mathbf{P}\{\Delta > z_\varepsilon\} = \varepsilon.$$

Using the same weak convergence we obtain the asymptotic normality of the MLE (see [6] or [8])

$$\hat{v}_n = \frac{\hat{\vartheta}_n - \vartheta_1}{\varphi_n} \implies \hat{v} = \Delta \mathbb{I}_{\{\Delta \geq 0\}}.$$

The limit behavior of the power functions we study under alternative $\vartheta_{u_*} = \vartheta_1 + u_*\varphi_n$. Let us fix $u_* > 0$.

We have the weak convergence of the likelihood ratio process under alternative too and therefore we can write

$$\begin{aligned} \sup_{v>0} Z_n(v) &= \sup_{v>0} \frac{L(\vartheta_1 + v\varphi_n, X^n)}{L(\vartheta_1, X^n)} = \frac{L(\vartheta_{u_*}, X^n)}{L(\vartheta_1, X^n)} \sup_{v>0} \frac{L(\vartheta_1 + v\varphi_n, X^n)}{L(\vartheta_{u_*}, X^n)} \\ &= \frac{L(\vartheta_{u_*}, X^n)}{L(\vartheta_1, X^n)} \sup_{v>0} \frac{L(\vartheta_{u_*} + (v - u_*)\varphi_n, X^n)}{L(\vartheta_{u_*}, X^n)}, \end{aligned}$$

where we followed the same lines as in the proof of the Lemma 2. Note that

$$\left(\frac{L(\vartheta_{u_*}, X^n)}{L(\vartheta_1, X^n)} \right)^{-1} = \frac{L(\vartheta_{u_*} - u_*\varphi_n, X^n)}{L(\vartheta_{u_*}, X^n)} \Rightarrow Z(-u_*) = \exp \left\{ -u_*\Delta - \frac{u_*^2}{2} \right\}$$

and

$$\frac{L(\vartheta_{u_*} + (v - u_*)\varphi_n, X^n)}{L(\vartheta_{u_*}, X^n)} \Rightarrow \exp \left\{ (v - u_*)\Delta - \frac{(v - u_*)^2}{2} \right\}.$$

Hence we obtain

$$\sup_{v>0} Z(v) \rightarrow \sup_{v>0} Z(v, u_*) = \sup_{v>0} \exp \left\{ v\Delta - \frac{(v^2 - 2vu_*)}{2} \right\}.$$

Therefore,

$$\begin{aligned} \beta(\psi_n^o, u_*) &\rightarrow \mathbf{P} \{ (\Delta + u_*) \mathbb{I}_{\{\Delta + u_* \geq 0\}} > z_\varepsilon \} = \mathbf{P} \{ \max[\Delta + u_*, 0] > z_\varepsilon \} \\ &= \mathbf{P} \{ \max[\Delta, -u_*] > z_\varepsilon - u_* \} \mathbb{I}_{\{z_\varepsilon \geq u_*\}} \\ &\quad + \mathbb{I}_{\{z_\varepsilon < u_*\}} \left[\mathbf{P} \{ \Delta < -u_* \} \mathbb{I}_{\{z_\varepsilon - u_* < -u_*\}} + \mathbf{P} \{ \Delta > z_\varepsilon - u_*, \Delta > -u_* \} \right] \\ &= \mathbf{P} \{ \Delta > z_\varepsilon - u_* \} = \beta^*(u_*). \end{aligned}$$

Similarly we have

$$\begin{aligned} \mathbf{P}_{\vartheta_{u_*}}^{(n)} \{ Q(X^n) > h_\varepsilon \} &\longrightarrow \mathbf{P}_{\vartheta_1} \{ (\Delta + u_*)^2 \mathbb{I}_{\{\Delta + u_* \geq 0\}} > z_\varepsilon^2 \} \\ &= \mathbf{P} \{ \Delta > z_\varepsilon - u_* \} = \beta^*(u_*). \end{aligned}$$

Therefore the tests are LAUMP. \square

This asymptotic equivalence and optimality of these tests is a well known property of the tests in regular statistical experiences (see, e.g. [12], [9]). We present these properties of the tests because we have to compare the asymptotics of these tests in regular and singular statistical models (see [4]). At particularly, we will see that the asymptotic properties of these tests in non regular situations will be quite different.

4.3 Bayesian tests

Suppose now that the unknown parameter ϑ is a random variable with *a priori* density $p(\theta)$, $\theta \in [\vartheta_1, b]$. Here $p(\cdot)$ is a known continuous function satisfying condition $p(\vartheta_1) > 0$. We consider two approach. The first one is based on the bayesian estimator and the second on the averaged likelihood ratio function.

Let us consider the Wald type test but based on bayesian estimator $\tilde{\vartheta}_n$

$$\tilde{\psi}_n(X^n) = \mathbb{I}_{\{\varphi_n^{-1}(\tilde{\vartheta}_n - \vartheta_1) > g_\varepsilon\}}.$$

Remind that the BE for quadratic loss function is

$$\tilde{\vartheta}_n = \int_{\vartheta_1}^b \theta p(\theta|X^n) d\theta = \frac{\int_{\vartheta_1}^b \theta p(\theta) L(\theta, \vartheta_1, X^n) d\theta}{\int_{\vartheta_1}^b p(\theta) L(\theta, \vartheta_1, X^n) d\theta}.$$

Let us change the variables in this integrals $\theta = \vartheta_1 + v\varphi_n$, then we obtain the relation

$$\frac{\tilde{\vartheta}_n - \vartheta_1}{\varphi_n} = \frac{\int_{\mathbb{U}_n^+} v p(\vartheta_1 + v\varphi_n) Z_n(v) dv}{\int_{\mathbb{U}_n^+} p(\vartheta_1 + v\varphi_n) Z_n(v) dv}.$$

The properties of $Z_n(\cdot)$ verified in the proof of the Theorem 1 allow to prove the following convergence in distribution (see [6] or [8])

$$\begin{aligned} \frac{\tilde{\vartheta}_n - \vartheta_1}{\varphi_n} &\Rightarrow \tilde{u} = \frac{\int_0^\infty v Z(v) dv}{\int_0^\infty Z(v) dv} \\ &= \frac{e^{\Delta^2/2} \int_0^\infty (v - \Delta) \exp\left\{-\frac{(v-\Delta)^2}{2}\right\} dv}{e^{\Delta^2/2} \int_0^\infty \exp\left\{-\frac{(v-\Delta)^2}{2}\right\} dv} + \Delta \\ &= \frac{-\exp\left\{-\frac{(v-\Delta)^2}{2}\right\} \Big|_{v=0}^{+\infty}}{\sqrt{2\pi} \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp\left\{-\frac{(v-\Delta)^2}{2}\right\} dv} + \Delta \\ &= \frac{\exp\left\{-\frac{\Delta^2}{2}\right\}}{\sqrt{2\pi} (1 - F(-\Delta))} + \Delta = \frac{f(\Delta)}{F(\Delta)} + \Delta, \end{aligned}$$

where $f(\cdot)$ and $F(\cdot)$ are the density and distribution function of the standard normal distribution. The similar calculation under alternatives allows us to

write the limit power function of $\tilde{\psi}_n$ as follows.

$$\begin{aligned}
\beta(\tilde{\psi}_n, u) &= \mathbf{P}_{\vartheta_1 + u\varphi_n} \left\{ \varphi_n^{-1} (\tilde{\vartheta}_n - \vartheta_u) + u > g_\varepsilon \right\} \\
&\longrightarrow \mathbf{P}_u \left\{ \frac{\int_{-u}^{\infty} v Z(v) dv}{\int_{-u}^{\infty} Z(v) dv} + u > g_\varepsilon \right\} \\
&= \mathbf{P}_u \left\{ \frac{\exp \left\{ -\frac{(\Delta+u)^2}{2} \right\}}{\sqrt{2\pi} F(\Delta+u)} + \Delta + u > g_\varepsilon \right\} \\
&= \mathbf{P}_u \left\{ \frac{f(\Delta+u)}{F(\Delta+u)} + \Delta + u > g_\varepsilon \right\}.
\end{aligned}$$

Another possibility in bayesian approach is to define the test as the test with the minimal mean error. Denote $\alpha(\bar{\psi}_n, \theta) = 1 - \beta(\bar{\psi}_n, \theta)$ the type two error under alternative and introduce the mean error

$$\alpha(\bar{\psi}_n) = \int_{\vartheta_1}^b \alpha(\bar{\psi}_n, \theta) p(\theta) d\theta.$$

The bayesian test $\tilde{\psi}_n(X^n)$ is defined as the test which minimizes the mean error

$$\alpha(\tilde{\psi}_n) = \inf_{\bar{\psi}_n \in \mathcal{K}_\varepsilon} \alpha(\bar{\psi}_n).$$

The integral we can write as follows

$$\begin{aligned}
\int_{\vartheta_1}^b (1 - \mathbf{E}_\theta \bar{\psi}_n(X^n)) p(\theta) d\theta &= \int_{\vartheta_1}^b \int (1 - \bar{\psi}_n(x^n)) d\mathbf{P}_\theta p(\theta) d\theta \\
&= \int (1 - \bar{\psi}_n(x^n)) d\tilde{\mathbf{P}} = \tilde{\mathbf{E}} (1 - \bar{\psi}_n(X^n)),
\end{aligned}$$

where we denoted \mathbf{P}_θ the distribution of the sample X^n and

$$\tilde{\mathbf{P}}(X^n \in A) = \int_{\vartheta_1}^b \mathbf{P}_\theta(X^n \in A) p(\theta) d\theta.$$

The average power $\beta(\tilde{\psi}_n) = \tilde{\mathbf{E}}^{(n)} \bar{\psi}_n(X^n)$ is the same as if we have two simple hypotheses. Under \mathcal{H}_1 we observe a Poisson process of intensity function $\lambda(\vartheta_1, \cdot)$, and under alternative \mathcal{H}_2 the observed point process has random intensity and its measure is $\tilde{\mathbf{P}}$. This process is a mixture (according to the density $p(\theta)$) of inhomogeneous Poisson processes with intensities $\lambda(\theta, \cdot)$,

$\theta \in [\vartheta_1, b]$. This means that we have two simple hypotheses and the most powerful test by Neyman-Pearson lemma is of the form

$$\tilde{\psi}_n = \mathbb{I}_{\{\tilde{L}(X^n) > \tilde{k}_\varepsilon\}}, \quad \mathbf{E}_{\vartheta_1} \tilde{\psi}_n(X^n) = \varepsilon,$$

where the likelihood ratio

$$\tilde{L}(X^n) = \frac{d\tilde{\mathbf{P}}}{d\mathbf{P}_{\vartheta_1}}(X^n) = \int_{\vartheta_1}^b \frac{d\mathbf{P}_\theta}{d\mathbf{P}_{\vartheta_1}}(X^n) p(\theta) d\theta.$$

To study this test under hypothesis we change the variables

$$\tilde{L}(X^n) = \int_{\vartheta_1}^b L(\theta, \vartheta_1, X^n) p(\theta) d\theta = \varphi_n \int_0^{\varphi_n^{-1}(b-\vartheta_1)} Z_n(v) p(\vartheta_1 + v\varphi_n) dv.$$

The limit of the last integral was already described above and this allow us to write

$$\begin{aligned} R_n(X^n) &= \frac{\tilde{L}(X^n)}{p(\vartheta_1) \varphi_n} = \frac{1}{p(\vartheta_1)} \int_0^{\varphi_n^{-1}(b-\vartheta_1)} e^{v\Delta_n - \frac{v^2}{2} + r_n} p(\vartheta_1 + v\varphi_n) dv \\ &\implies \int_0^\infty e^{v\Delta - \frac{v^2}{2}} dv = e^{\frac{\Delta^2}{2}} \int_{-\Delta}^\infty e^{-\frac{y^2}{2}} dy = \sqrt{2\pi} e^{\frac{\Delta^2}{2}} (1 - F(-\Delta)) = \frac{F(\Delta)}{f(\Delta)}, \end{aligned}$$

where $F(\cdot)$ and $f(\cdot)$ are the distribution function and density of the standard gaussian random variable Δ . Hence if we take k_ε as solution of the equation

$$\mathbf{P} \left\{ \frac{F(\Delta)}{f(\Delta)} > k_\varepsilon \right\} = \varepsilon, \quad (6)$$

then the test $\tilde{\psi}_n(X^n) = \mathbb{I}_{\{R_n > k_\varepsilon\}}$ belongs to \mathcal{K}_ε . The similar calculation provides us the limit power function

$$\mathbf{P}_{\vartheta_{u_*}} \{R_n > k_\varepsilon\} \longrightarrow \mathbf{P} \left\{ \frac{F(\Delta + u_*)}{f(\Delta + u_*)} > k_\varepsilon \right\}.$$

5 Simulations

Below we present the results of numerical simulations of the power functions of the tests. We observe n independent realizations $X_j = \{X_j(t), t \in [0, 3]\}$; $j = 1, \dots, n$ of inhomogeneous Poisson process of intensity function

$$\lambda(\vartheta, t) = 3 \cos^2(\vartheta t) + 1, \quad 0 \leq t \leq 3, \quad \vartheta \in [3, 7].$$

where $\vartheta_1 = 3$. The Fisher information at the point ϑ_1 is $I(\vartheta_1) \approx 19.82$. Recall that all tests (except bayesian tests) in regular case are LAUMP. Therefore they have the same limit power function. Our goal is to study the power functions of different tests for finite n .

For the normalized likelihood ratio $Z_n(u)$ we have the expression :

$$Z_n(u) = \exp \left\{ \varphi_n \sum_{j=1}^n \int_0^3 \ln \frac{3 \cos^2((3 + u\varphi_n)t) + 1}{3 \cos^2(3t) + 1} dX_j(t) - \frac{3n}{4(3 + u\varphi_n)} \sin(6(3 + u\varphi_n)) + \frac{n}{4} \sin(18) \right\},$$

where $\varphi_n = (19.82n)^{-1/2}$.

The simulation of the observations allows us to obtain the power functions presented on the following pictures.

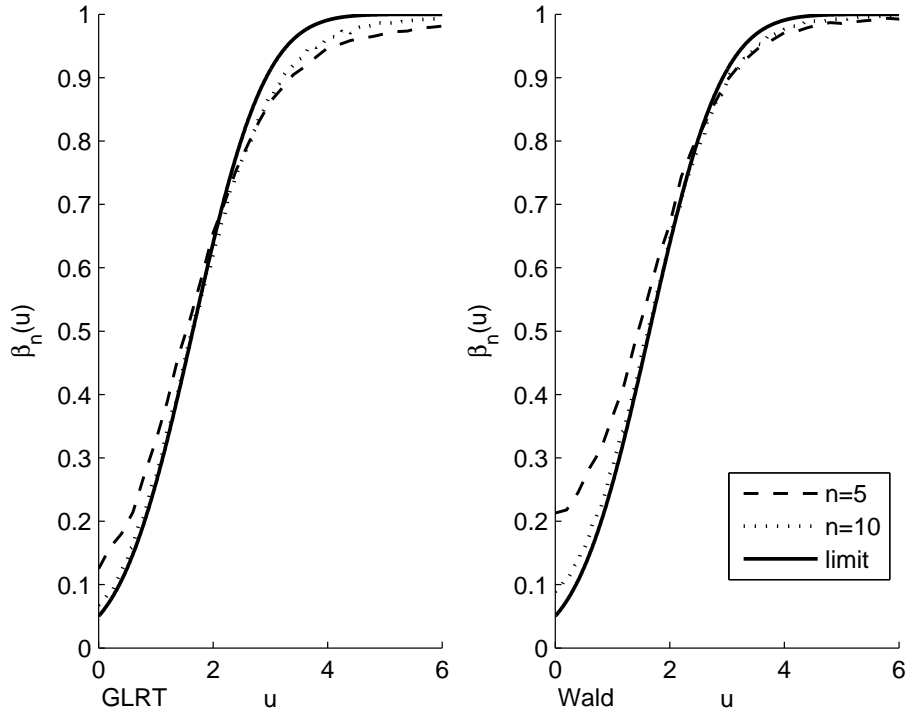


Figure 1: Power functions of GLRT and Wald's test

The calculation of the numerical values of the power function of the SFT was done as follows. We define an increasing sequence of u beginning at $u = 0$.

Then for every u , we simulate N i.i.d observations of inhomogeneous Poisson processes $Y_i = X^{n,i}$, $i = 1, \dots, N$ with the intensity function $\lambda(3 + u\varphi_n, t)$ and calculate the corresponding statistics $\Delta_{n,i}(3, Y_i)$, $i = 1, \dots, N$. The empirical frequency of acceptance of the alternative gives us the estimate of the power function

$$\beta_n(u) \approx \frac{1}{N} \sum_{i=1}^N \mathbb{I}_{\{\Delta_{n,i}(3, Y_i) > z_\varepsilon\}}.$$

We repeat this procedure for different values of u up to the value of $\beta^*(u)$ close to 1.

In the calculation of the power function of the Bayesian test(BT1), we take as the density *a priori* the uniform distribution $p(\vartheta) \sim \mathcal{U}([3, 7])$. The thresholds of the BT1 are obtained by simulating $M = 10^5$ r.v.'s. $\Delta_i \sim \mathcal{N}(0, 1)$, $i = 1, \dots, M$, calculating for each of them the quantity

$$\frac{f(\Delta_i)}{F(\Delta_i)} + \Delta_i, \quad i = 1, \dots, 10^5$$

and taking $(1 - \varepsilon) 10^5$ -th greatest between them.

ε	0.01	0.05	0.10	0.2	0.4	0.5
g_ε	2.325	1.751	1.478	1.193	0.895	0.794

Table 1: The thresholds of the BT1.

Note that for the small values of n , under alternative, one can see that the power function of SFT starts to decrease. This interesting fact can be explain by the strongly non linear dependence of the likelihood ratio of the parameter. The test statistics $\Delta_n = \Delta_n(3, X^n)$ under alternative can be written as follows

$$\begin{aligned} \Delta_n &= \varphi_n \sum_{j=1}^n \int_0^T \frac{\dot{\lambda}(\vartheta_1, t)}{\lambda(\vartheta_1, t)} [dX_j(t) - \lambda(\vartheta_1 + u\varphi_n, t) dt] \\ &\quad + \sqrt{\frac{n}{I(\vartheta_1)}} \int_0^T \frac{\dot{\lambda}(\vartheta_1, t)}{\lambda(\vartheta_1, t)} [\lambda(\vartheta_1 + u\varphi_n, t) - \lambda(\vartheta_1, t)] dt \\ &= -3\varphi_n \sum_{j=1}^n \int_0^3 \frac{t \sin(6t)}{3 \cos^2(3t) + 1} [dX_j(t) - (3 \cos^2((3 + u\varphi_n)t) + 1) dt] \\ &\quad + 9 \sqrt{\frac{n}{I(\vartheta_1)}} \int_0^3 \frac{t \sin(6t)}{3 \cos^2(3t) + 1} \times [\cos^2(3t) - \cos^2((3 + u\varphi_n)t)] dt. \end{aligned}$$

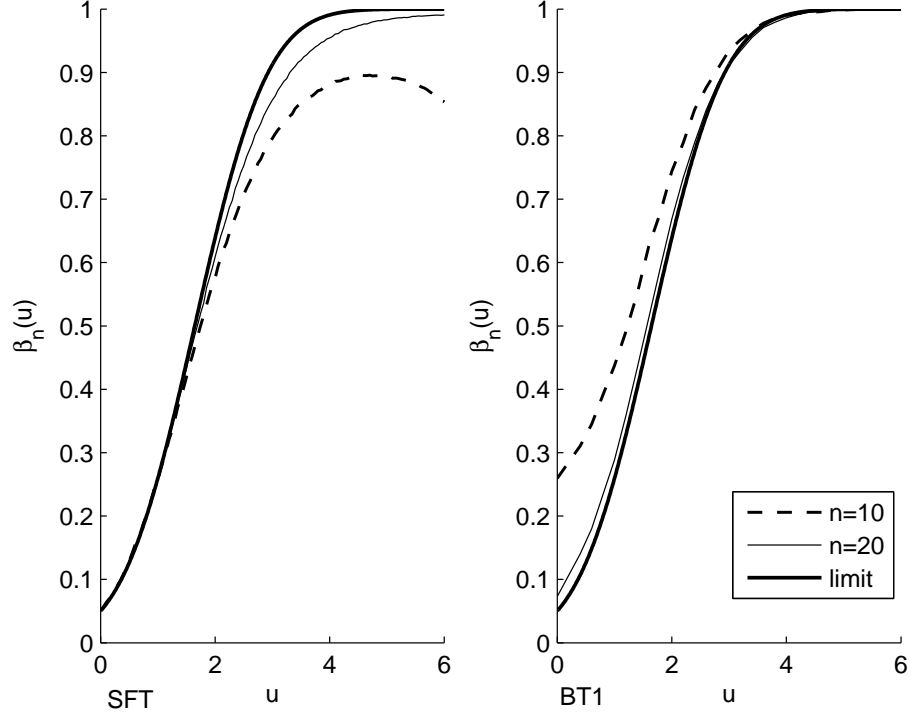


Figure 2: Power functions of SFT and BT1 in regular case

The last integral in the r.h.s. of this equation for some values of u becomes negative, and this leads to decreasing of the power function of SFT for the value $n = 10$.

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